A stochastic optimal velocity model and its long-lived metastability

Masahiro Kanai

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8914, Japan.

Katsuhiro Nishinari

Department of Aeronautics and Astronautics, Faculty of Engineering,
University of Tokyo, 7-3-1 Hongo, Tokyo 113-8656, Japan.

Tetsuji Tokihiro

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8914, Japan.
(Dated: May 23, 2009)

In this paper, we propose a stochastic cellular automaton model of traffic flow extending two exactly solvable stochastic models, i.e., the asymmetric simple exclusion process and the zero range process. Moreover it is regarded as a stochastic extension of the optimal velocity model. In the fundamental diagram (flux-density diagram), our model exhibits several regions of density where more than one stable state coexists at the same density in spite of the stochastic nature of its dynamical rule. Moreover, we observe that two long-lived metastable states appear for a transitional period, and that the dynamical phase transition from a metastable state to another metastable/stable state occurs sharply and spontaneously.

Traffic dynamics has been attracting much attention from physicists, engineers and mathematicians as a typical example of non-equilibrium statistical mechanics of self-driven many particle systems for the last decade [1, 2, 3]. Statistical properties of traffic phenomena are studied empirically by using the *fundamental diagram*, which displays the relation of the flux (the average velocity of vehicles multiplied by the density of them) to the density. We have been found an emergence of more than one different flux at the same density in the transit region from free to congested phase is almost universal in real traffic [4]. Recent experimental studies also show that the phase transition occurs discontinuously against the density and structurally complex states appear around the critical density [5, 6, 7].

While traffic models with many parameters and complex rules may reproduce empirical data, a strong mathematical support, if any, allows a direct connection between microscopic modelling and the universal feature extracted from various traffic flows. In this paper, we therefore propose a cellular automaton (CA) model extending two significant stochastic processes, i.e., the asymmetric simple exclusion process (ASEP) and the zero range process (ZRP) as described later. Moreover, we find that our model is supported by a successful traffic model, *the optimal velocity (OV) model* [8, 9]. The OV model, which is a continuous and deterministic model, is expressed by coupled differential equations; $d^2x_i/dt^2 = a[V(x_{i+1} - x_i) - dx_i/dt]$, where $x_i = x_i(t)$ is the position of the i-th vehicle at time t and the function V is called the optimal velocity function, which gives the optimal speed of a vehicle according to its headway $x_{i+1} - x_i$ (the i-th vehicle follows the $(i+1)$-th in the same lane). The intrinsic parameter a indicates the driver's sensitivity to traffic situations and governs the stability of flow. Our stochastic CA model is, however, different from a noisy OV model [10] as described below.

First of all, we explain the general framework of our stochastic CA model for one-lane traffic. N vehicles are moving on a single-lane road which is divided into a one-dimensional array of L sites. Each site contains one vehicle at most, and collision and overtaking are thus prohibited (the so-called *hard-core exclusion rule*). In this paper, parallel updating is adopted, i.e., all the vehicles attempt to move at each step. We introduce a probability distribution function $w_i^t(m)$ which gives the probability, or the driver's *intention*, of the i-th vehicle hopping m ($m = 0, 1, 2, \ldots$) sites ahead at time t. Then assuming that the next intention $w_i^{t+1}(m)$ is determined by a function f_i depending on $w_i^t(0), w_i^t(1), \ldots$ and the positions $x_1^t, x_2^t, \ldots, x_N^t$, the configuration of vehicles is recursively updated according to the following procedure:

- For each vehicle, calculate the next intention to hop m sites with the configuration $x_1^t, x_2^t, \ldots, x_N^t$ and the intention $w_i^t(0), w_i^t(1), \ldots$;

$$w_i^{t+1}(m) = f_i\big(w_i^t(0), w_i^t(1), \ldots; x_1^t, \ldots, x_N^t; m\big). \quad (1)$$

- Determine the number of sites V_i^{t+1} at which a vehicle moves (i.e. the velocity) probabilistically according to the intention w_i^{t+1}. In other words, the probability of $\mathsf{V}_i^{t+1} = m$ is equal to $w_i^{t+1}(m)$.

- Each vehicle moves avoiding a collision;

$$x_i^{t+1} = x_i^t + \min(\Delta x_i^t, \mathsf{V}_i^{t+1}), \quad (2)$$

where $\Delta x_i^t = x_{i+1}^t - x_i^t - 1$ defines the headway, and the vehicles thus move at either V_i^{t+1} or Δx_i^t sites. If $\mathsf{V}_i^{t+1} > \Delta x_i^t$, a vehicle must stop at the cell $x_{i+1}^t - 1$.

In what follows, we assume that the maximum allowed velocity is equal to 1, i.e., if $m \geqslant 2$, $w_i^t(m) = 0$. Then,
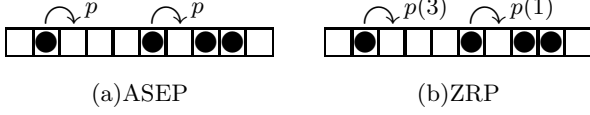
(a)ASEP   (b)ZRP

FIG. 1: Schematic view of the tagged-particle model for ASEP(a) and ZRP(b). The hopping probability of a particle depends on the gap size in front of it in ZRP, while it is always constant in ASEP. In both cases, hopping to an occupied cell is prohibited.

putting $v_i^t \equiv w_i^t(1)$ (accordingly $w_i^t(0) = 1 - v_i^t$), we propose a special type of f_i in (1) as

$$v_i^{t+1} = (1 - a_i)v_i^t + a_i V_i(\Delta x_i^t) \qquad (\forall t \geq 0, \ \forall i), \quad (3)$$

where a_i $(0 \leq a_i \leq 1)$ is a parameter and the function V_i is restricted to values in the interval $[0, 1]$ so that v_i^t also should be within $[0, 1]$. The intrinsic parameter a_i, a weighting factor of the optimal velocity $V_i(\Delta x_i^t)$ to the intention w_i^{t+1}, corresponds to the driver's sensitivity to a traffic condition. As long as the vehicles move separately, we can also rewrite (2) simply as $x_i^{t+1} = x_i^t + 1$ with probability v_i^{t+1}. (Note that the original OV model does not support the hard-core exclusion [8, 9].) Therefore, v_i^t can be regarded as the average velocity, i.e., $\langle x_i^{t+1} \rangle = \langle x_i^t \rangle + v_i^{t+1}$ in the sense of expectation values. We call the model expressed by (3) *the stochastic optimal velocity (SOV) model* because of its formal similarity to a discrete version of the OV model (or a coupled map lattice[11])

$$x_i^{t+1} = x_i^t + v_i^{t+1}\Delta t, \qquad (4)$$
$$v_i^{t+1} = (1 - a\Delta t)v_i^t + (a\Delta t)V(\Delta x_i^t), \qquad (5)$$

where Δt is a time interval and the OV model is recovered in the limit $\Delta t \to 0$. In this special case that the maximum allowed velocity equals to 1, we thus have an obvious correspondence of our stochastic CA model to an existent traffic model. As a matter of convenience, we set $a_i = a$ and $V_i = V$ $(\forall i)$ hereafter.

From the viewpoint of mathematical interest, the SOV model includes two significant stochastic models. When $a = 0$, (3) becomes $v_i^{t+1} = v_i^t$, i.e., the model reduces to ASEP [12, 13, 14, 15] with a constant hopping probability $p \equiv v_i^0$. When $a = 1$, (3) becomes $v_i^{t+1} = V(\Delta x_i^t)$, i.e., the model reduces to ZRP [16] considering the headways $\{\Delta x_i^t\}$ as the stochastic variables of ZRP. In ZRP, the hopping probability of a vehicle is determined exclusively by its present headway. Fig.1 illustrates the two stochastic models schematically. ASEP and ZRP are both known to be *exactly solvable* in the sense that the probability distribution of the configuration of vehicles in the stationary state can be exactly calculated [17, 18], and thus our model admits an exact calculation of the fundamental diagram in the special cases.

In order to investigate a phenomenological feature, we take a realistic form of the OV function as

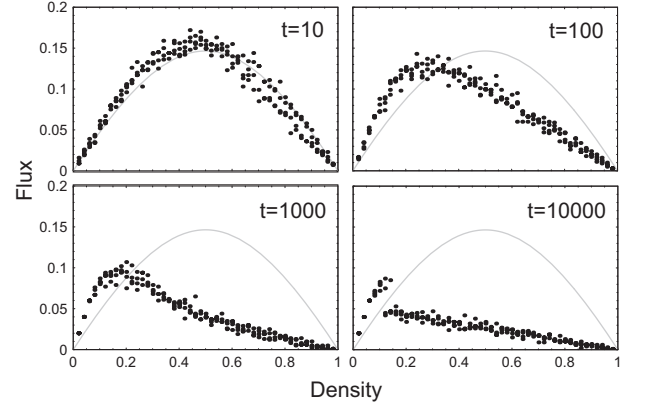$$V(x) = \frac{\tanh(x - c) + \tanh c}{1 + \tanh c}, \qquad (6)$$

FIG. 2: The fundamental diagram of the SOV model with the OV function (6) (c=1.5 and $a = 0.01$) plotted at each time stage t, starting from uniform/random states with $p (\equiv v_i^0) = 0.5$, including the exact curve(gray) of ASEP for comparison [13]. The system size is $L = 1000$.

which was investigated in [8]. We find that the fundamental diagrams simulated with (6) have a quantitative agreement with the exact calculation of ZRP ($a = 1$) up to $a \sim 0.6$. In contrast, the fundamental diagram of the SOV model does not come closer to that of ASEP as a approaches to 0 although the SOV model coincides with ASEP at $a = 0$. Fig. 2 shows that a curve similar to the diagram of ASEP appears only for the first few steps ($t = 10$) and then changes the shape rapidly ($t = 100, 1000$). Surprisingly, when the diagram becomes stationary, it allows a discontinuous point and two overlapping stable states around the density $\rho \sim 0.14$ ($t = 10000$).

Let us study the discontinuity of the flux in detail. Fig. 3 shows the fundamental diagram expanded around the discontinuous point. We have three distinct branches and then call them as follows: *free-flow phase* (vehicles move without interactions), *congested phase* (a mixture of small clusters and free vehicles), and *jam phase* (one big stable jam transmitting backward). These branches appear in the fundamental diagram, respectively as a segment of line with slope 1(free-flow), as a thick curve with a slight positive slope(congested), and as a thick line with a negative slope(jam). Note that the congested and jam lines shows some fluctuations due to a randomness of the SOV model. Fig. 3(a) shows snapshots of the flux at $t = 1000$ and 5000. There exists midstream flux between free-flow and congested phases at $t = 1000$, and between congested and jam phases at $t = 5000$. This suggests that the high-density free-flow states can hold until $t = 1000$ but not until $t = 5000$, and that the high-density congested states have already started to decay into jam states before $t = 5000$. We have thereby revealed the existence of two metastable branches leading out of the free-flow or congested lines. Comparing Fig.3(a) with Fig. 3(b), we have six qualitatively distinct regions of density(Fig.3(b)); *free region* S_1 (includ-
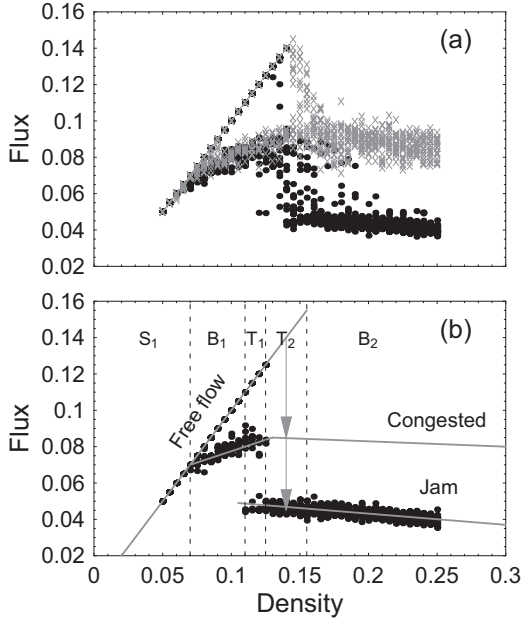
FIG. 3: (a)The expanded fundamental diagram of the SOV model with $a = 0.01$ at $t = 1000$ (gray) and $t = 5000$ (black) starting from two typical states; the uniform state with equal spacing of vehicles and $p \, (\equiv v_i^0) = 1$, and the random state with random spacing and $p = 1$. We observe three distinct branches, which we call the free-flow, congested, and jam branch. They survive even in the stationary state, which is plotted at $t = 50000$ in (b). (b)The stationary states (black) and the averaged three branches (gray lines) are plotted at $t = 50000$. The vertical dotted lines distinguish the regions of density from S_1 to B_2. The arrows in T_2 indicate the trace of a metastable free-flow state decaying to the lower branches. (see also Fig. 4).

ing only free-flow phase), *bistable region* B_1 (including free-flow and congested phases, which are both stable), *tristable region* T_1 (including all the three phases, which are all stable), *tristable region* T_2 (including free-flow, congested and jam phases. The former two phases are stable and the last one is metastable), *bistable region* B_2 (including congested and jam phases. The former is metastable, and the latter is stable), and *jam region* S_2 (including only stable jam phase, which is not displayed here). We stress that the tristable region T_1 is a novel and remarkable characteristic of many-particle systems, and that the successive phase transitions from a free-flow state to a jam state via a congested state occur respectively on a short time scale.

Let us study the *dynamical phase transition* especially at the density $\rho = 0.14$, indicated by successive arrows in Fig. 3(b). Fig. 4(upper) shows the flux plotted against time. It is striking that there appear three plateaus which respectively correspond to a free-flow state, a congested state, and a jam state, and that the flux changes sharply from one plateau to another. In other words, the metastable states have a remarkably long lifetime before undergoing a sudden phase transition. Stochastic mod-
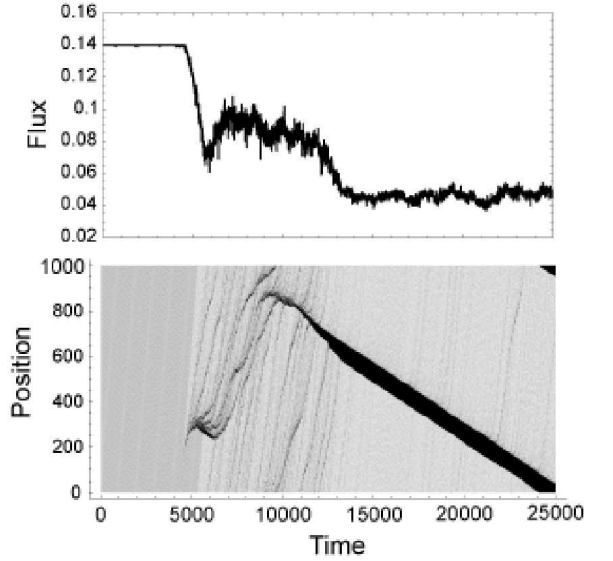

FIG. 4: The time evolution of flux at the density $\rho = 0.14$ starting from the uniform state. We observe two plateaus at the flux $Q = 0.14$ with a lifetime $T \simeq 5000$, and $Q \simeq 0.08$ with $T \simeq 7000$ before reaching the stationary jam state(upper). The lower figure shows the corresponding spatio-temporal diagram, where vehicles (black dots) move from bottom up. (Note that the periodic boundary condition is imposed.)

els, in general, are not anticipated to have such long-lived metastable states because stochastic fluctuations break a stability of states very soon [7]. Fig. 4(lower) shows the spatio-temporal diagram corresponding to the dynamical phase transition. Starting from a free-flow state, the uniform configuration stochastically breaks down at time $t \sim 5000$, and then the free-flow state is rapidly replaced by a congested state where a lot of clusters are forming and dissolving, moving forward and backward. Fig. 5 shows the distribution of headways at several time stages. We find that the distribution of headways changes significantly after each phase transition. In particular, the ratio of vehicles with null headway increases. As for a congested state, it is meaningful to evaluate the average size of clusters from a distribution of headways. If the ratio of the vehicles with null headway is b_0, the average cluster size ℓ can be evaluated as $1/(1-b_0)$. In the present case, the average size of clusters is about $1.2 \sim 1.4$ during $t = 6000 \sim 12000$. We also have some remarks, from a microscopic viewpoint, on a single cluster: Since the sensitivity parameter a is set to a small value, the intention v_i^t does not change a lot before the vehicle catches up with the tail of a cluster(i.e. $v_i^t \sim 1$). Accordingly, the aggregation rate α is roughly estimated at the density of free region behind the cluster; $\alpha \sim (1 - b_0)\rho$. In the present case, the aggregation rate averaged over the whole clusters of a congested state is $0.10 \sim 0.12$. For the same reason, we can estimate the average velocity of the front vehicle of a cluster at $(1 - a)^{\ell/\delta}$, where δ denotes the dissolution rate and ℓ/δ indicates the duration of cap-
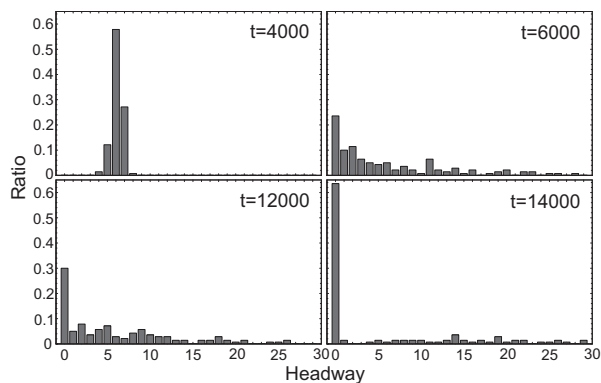
FIG. 5: The distribution of the headways with which the vehicles move at each time stage t. It changes a lot after phase transitions($t \sim 5000, 12000$). The traffic states, free-flow($t < 5000$), congested($5000 < t < 12000$), and jam($t > 12000$) respectively show their specific pictures.

ture. Since δ is also equivalent to the average velocity of the front vehicle, it amounts roughly to $1 - a\ell$ after all. In the present case, the dissolution rate averaged over the whole clusters of a congested state is $0.86 \sim 0.88$. Then, the average lifetime of the clusters $\ell/(\delta - \alpha)$ is estimated at $1.54 \sim 1.89$. The above estimations are appropriate only when clusters are small and spaced apart. As many clusters arise everywhere and gather, a vehicle out of a cluster tends to be caught in another cluster again before it recovers its intention at full value. Consequently,

the clusters arising nearby reduce their dissolution rates, and finally grow into a jam moving steadily backward. The steady transmitting velocity (the aggregation rate) is estimated at -0.055, coinciding with Fig. 4.

In this paper, beginning with a general scheme, we have proposed a stochastic CA model to which we introduce a probability distribution function of the vehicle's velocity. It includes two exactly solvable stochastic processes, and it is also regarded as a stochastic generalization of the OV model. Moreover, it exhibits the following features: In spite of a stochastic model, the fundamental diagram shows that there coexist two or three stable phases (free-flow, congested, and jam) in a region of density. As the density increases, the free-flow and congested states lose stability and change into metastable states which can be observed only for a transitional period. Moreover the dynamical phase transition from a metastable state to another metastable/stable state, which is triggered by stochastic perturbation, occurs sharply and spontaneously. We consider that the metastable state may be relevant to the transient congested state observed in the upstream of on-ramp [5, 19]. Such a dynamical phase transition has not been observed in previous works [8, 9, 10] or among existent many-particle systems. Further studies, e.g., on another choice of the OV function, under open boundary conditions, and on the general (i.e. multi-velocity) version will be given in subsequent publications [20].

[1] D. Helbing, Rev. Mod. Phys. **73**, 1067 (2001).
[2] D. Chowdhury, L. Santen and A. Schadschneider, Phys. Rep.**329**, 199 (2000).
[3] T. Nagatani, Rep. Prog. Phys. **65**, 1331 (2002).
[4] K. Nishinari and M. Hayashi, *Traffic statistics in Tomei express way*, (The Mathematical Society of Traffic Flow, 1999, Nagoya).
[5] B. S. Kerner, *The Physics of Traffic*, Springer-Verlag, Berlin, New York 2004.
[6] M. Treiber, A. Hennecke and D. Helbing, Phys. Rev. E, **62**, 1805 (2000).
[7] K. Nishinari, M. Fukui and A. Schadschneider, J. Phys. A **37**, 3101 (2004).
[8] M. Bando, K. Hasebe, A. Nakayama, A. Shibata and Y. Sugiyama, Phys. Rev. E **51**, 1035 (1995).
[9] M. Bando, K. Hasebe, K. Nakanishi, A. Nakayama, A. Shibata and Y. Sugiyama, J. Phys. I France **5**, 1389 (1995).
[10] D. Helbing and M. Schreckenberg, Phys. Rev. E **59**,

R2505 (1999).
[11] S. Yukawa and M. Kikuchi, J. Phys. Soc. Jpn. **64**, 35 (1995).
[12] B. Derrida, E. Domany, and D. Mukamel, J. Stat. Phys. **69**, 667 (1992).
[13] B. Derrida, M. R. Evans, V.Hakim, and V.Pasquier, J. Phys. A **26**, 1493 (1993).
[14] N. Rajewsky, L. Santen, A. Schadschneider, and M. Schreckenberg, J. Stat. Phys. **92**, 151 (1998).
[15] G. M. Shütz, J. Phys. A **36**, R339 (2003).
[16] F. Spitzer, Adv. Math. **5**, 246 (1970).
[17] M. R. Evans, J. Phys. A **30**, 5669 (1997).
[18] O. J. O'Loan, M. R. Evans, and M. E. Cates, Phys. Rev. E **58**, 1404 (1998).
[19] N. Mitarai and H. Nakanishi, Phys. Rev. Lett. **85**, 1766 (2000).
[20] M. Kanai, K. Nishinari and T. Tokihiro, to be published.